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HIGH FREQUENCY VIBRATIONS OF QUARTZ PLATES BY EXPANSION IN SERIES OF EKSTEIN FUNCTIONS†

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Abstract—A solution is presented for high frequency vibrations of anisotropic, elastic plates applicable to rotated-Y-cuts of quartz with a pair of free edges. The solution is based on an expansion in a series of Ekstein's exact, normal functions for the infinite plate—retaining the first three terms: flexure, thickness-shear, face-shear.

1. INTRODUCTION

There is no difficulty, today, in obtaining exact solutions of the three-dimensional, linear equations of elasticity for problems of vibrations of infinite plates with a pair of free faces. For the isotropic plate, this was first accomplished by Rayleigh in 1889[1]; and Ekstein, in 1945[2], gave a solution for an anistropic plate applicable to the technologically important rotated-Y-cuts of quartz. Although many years passed before the resulting frequency equations were analyzed in detail, their properties are now well understood.

At a free edge of a plate with free faces, each of the infinity of modes of the infinite plate reflects, except for a few special cases, as an infinity of modes; and therein lies the difficulty with finding solutions for plates with free faces and even only a single free edge or pair of parallel free edges. The classical method of overcoming the difficulty has been the employment of approximate, two-dimensional equations of motion which already satisfy, at least approximately, the conditions of free faces so that the conditions for free edges are analogous to those for free faces in the three-dimensional case. A solution of the approximate equations for the infinite plate contains, usually, only a small number of modes approximating the corresponding ones of the infinity of modes from the three-dimensional equations. Each of the approximate modes reflects, at a free edge, at most as the total, small number; so that the frequency equation is finite. The approximate equations are generally obtained, especially in the case of the high frequency range, by expanding the three-dimensional displacements or strains or stresses in series of powers or polynomials or trigonometric functions of the thickness coordinate of the plate, followed by an averaging across the thickness which eliminates one of the three independent spatial variables from the equations of motion. Thus, all variations across the thickness, throughout the plate, are assumed to be of simple form and then are averaged. An alternative is to expand in a series of the exact functions, obtained from the exact solution of the three-dimensional equations for the infinite plate, and postpone all approximations to the boundary conditions at the edges. At that stage, only a small number of the exact functions may be retained. Then averaging over the thickness is performed only at the edges as the last step. This procedure was carried out in detail in a previous paper[3], retaining only the first two exact modes (flexure and thickness-shear) of the infinite plate. Although that application was to the AT-cut of quartz, the coupling with the face-shear mode was omitted in order to compare with an earlier approximation. The coupling with the face-shear is included in the present application.

2. EKSTEIN FUNCTIONS

We consider, first, vibrations of an infinite plate. In an x_i, $i = 1,2,3$ rectangular coordinate system, the plate is bounded by free surfaces $x_2 = \pm h$ and vibrates in modes of motion with

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straight crests along x_3. Anticipating that there will be n modes of propagation ($n = 1, 2, \ldots, \infty$), along the plate, each of which contains three wave lengths across the thickness, we may write, for the components of displacement essentially antisymmetric with respect to the middle of the plate:

$$
\begin{aligned}
u_1 &= \sum_{n=1}^{\infty} \sum_{i=1}^{3} A_{in} \cos \xi_n x_1 \sin \eta_{in} x_2, \\
u_2 &= \sum_{n=1}^{\infty} \sum_{i=1}^{3} B_{in} \sin \xi_n x_1 \cos \eta_{in} x_2, \\
u_3 &= \sum_{n=1}^{\infty} \sum_{i=1}^{3} C_{in} \sin \xi_n x_1 \cos \eta_{in} x_2.
\end{aligned}
\tag{1}
$$

With displacements of the form (1), the equations of motion for rotated-Y-cuts of quartz [4] reduce to

$$
\begin{aligned}
&c_{11}u_{1,11} + c_{66}u_{1,22} + \rho\omega^2 u_1 + (c_{12} + c_{66})u_{2,12} + (c_{14} + c_{56})u_{3,12} = 0, \\
&(c_{12} + c_{66})u_{1,12} + c_{66}u_{2,11} + c_{22}u_{2,22} + \rho\omega^2 u_2 + c_{56}u_{3,11} + c_{24}u_{3,22} = 0, \\
&(c_{14} + c_{56})u_{1,12} + c_{56}u_{2,11} + c_{24}u_{2,22} + c_{55}u_{3,11} + c_{44}u_{3,22} + \rho\omega^2 u_3 = 0.
\end{aligned}
\tag{2}
$$

Upon substituting (1) in (2), we find

$$
\begin{aligned}
&(\bar{c}_{11}\bar{\xi}_n^2 + \bar{\eta}_{in}^2 - \Omega^1)A_{in} + (1 + \bar{c}_{12})\bar{\xi}_n\bar{\eta}_{in}B_{in} + (\bar{c}_{14} + \bar{c}_{56})\bar{\xi}_n\bar{\eta}_{in}C_{in} = 0, \\
&(1 + \bar{c}_{12})\bar{\xi}_n\bar{\eta}_{in}A_{in} + (\bar{\xi}_n^2 + \bar{c}_{22}\bar{\eta}_{in}^2 - \Omega^2)B_{in} + (\bar{c}_{56}\bar{\xi}_n^2 + \bar{c}_{24}\bar{\eta}_{in}^2)C_{in} = 0, \\
&(\bar{c}_{14} + \bar{c}_{56})\bar{\xi}_n\bar{\eta}_{in}A_{in} + (\bar{c}_{56}\bar{\xi}_n^2 + \bar{c}_{24}\bar{\eta}_{in}^2)B_{in} + (\bar{c}_{55}\bar{\xi}_n^2 + \bar{c}_{44}\bar{\eta}_{in}^2 - \Omega^2)C_{in} = 0,
\end{aligned}
\tag{3}
$$

where

$$
\bar{\xi}_n = 2\xi_n h/\pi, \quad \bar{\eta}_{in} = 2\eta_{in}h/\pi, \quad \bar{c}_{pq} = c_{pq}/c_{66}, \quad \Omega^2 = 4h^2\rho\omega^2/\pi c_{66}.
\tag{4}
$$

$\bar{\xi}_n$ and $\bar{\eta}_{in}$ are the ratios of the thickness of the plate to the half wave lengths along and through the plate, respectively; Ω is the ratio of the frequency to the cut-off frequency of the fundamental thickness-shear mode. From (3),

$$
\begin{vmatrix}
\bar{c}_{11}\bar{\xi}_n^2 + \bar{\eta}_{in}^2 - \Omega^2 & 1 + \bar{c}_{12} & (\bar{c}_{14} + \bar{c}_{56})\bar{\xi}_n\bar{\eta}_{in} \\
1 + \bar{c}_{12} & \bar{\xi}_n^2 + \bar{c}_{22}\bar{\eta}_{in}^2 - \Omega^2 & \bar{c}_{56}\bar{\xi}_n^2 + \bar{c}_{24}\bar{\eta}_{in}^2 \\
(\bar{c}_{14} + \bar{c}_{56})\bar{\xi}_n\bar{\eta}_{in} & \bar{c}_{56}\bar{\xi}_n^2 + \bar{c}_{24}\bar{\eta}_{in}^2 & \bar{c}_{55}\bar{\xi}_n^2 + \bar{c}_{44}\bar{\eta}_{in}^2 - \Omega^2
\end{vmatrix} = 0
\tag{5}
$$

and $B_{in}/A_{in} = \beta_{in}$, say, and $C_{in}/A_{in} = \gamma_{in}$, say, so that

$$
\begin{aligned}
\beta_{in} &= \frac{(\Omega^2 - \bar{c}_{11}\bar{\xi}_n^2 - \bar{\eta}_{in}^2)(\bar{c}_{56}\bar{\xi}_n^2 + \bar{c}_{24}\bar{\eta}_{in}^2) + (1 + \bar{c}_{12})(\bar{c}_{14} + \bar{c}_{56})\bar{\xi}_n^2\bar{\eta}_{in}^2}{\bar{\xi}_n\bar{\eta}_{in}[(1 + \bar{c}_{12})(\bar{c}_{56}\bar{\xi}_n^2 + \bar{c}_{24}\bar{\eta}_{in}^2) - (\bar{c}_{14} + \bar{c}_{56})(\bar{\xi}_n^2 + \bar{c}_{22}\bar{\eta}_{in}^2 - \Omega^2)]}, \\
\gamma_{in} &= \frac{(\Omega^2 - \bar{c}_{11}\bar{\xi}_n^2 - \bar{\eta}_{in}^2)(\Omega^2 - \bar{\xi}_n^2 - \bar{c}_{22}\bar{\eta}_{in}^2) - (1 + \bar{c}_{12})^2\bar{\xi}_n^2\bar{\eta}_{in}^2}{\bar{\xi}_n\bar{\eta}_{in}[(1 + \bar{c}_{12})(\bar{c}_{56}\bar{\xi}_n^2 + \bar{c}_{24}\bar{\eta}_{in}^2) - (\bar{c}_{14} + \bar{c}_{56})(\bar{\xi}_n^2 + \bar{c}_{22}\bar{\eta}_{in}^2 - \Omega^2)]}
\end{aligned}
\tag{6}
$$

The components of traction on $x_2 = \pm h$ are given by

$$
\begin{aligned}
T_{21} &= c_{56}u_{3,1} + c_{66}(u_{2,1} + u_{1,2}),\\
T_{22} &= c_{12}u_{1,1} + c_{22}u_{2,2} + c_{24}u_{3,2},\\
T_{23} &= c_{14}u_{1,1} + c_{24}u_{2,2} + c_{44}u_{3,2},
\end{aligned}
\tag{7}
$$

whence, on $x_2 = \pm h$,

$$
\begin{aligned}
T_{21} &= \sum_{n=1}^{\infty}\sum_{i=1}^{3} A_{in}\lambda_{in} \cos \xi_n x_1 \cos \eta_{in} h,\\
T_{22} &= \mp \sum_{n=1}^{\infty}\sum_{i=1}^{3} A_{in}\mu_{in} \sin \xi_n x_1 \sin \eta_{in} h,\\
T_{23} &= \mp \sum_{n=1}^{\infty}\sum_{i=1}^{3} A_{in}\nu_{in} \sin \xi_n x_1 \sin \eta_{in} h,
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
\lambda_{in} &= c_{66}\eta_{in} + c_{66}\xi_n\beta_{in} + c_{56}\xi_n\gamma_{in}\\
\mu_{in} &= c_{21}\xi_n + c_{22}\eta_{in}\beta_{in} + c_{24}\xi_{in}\gamma_{in},\\
\nu_{in} &= c_{14}\xi_n + c_{24}\eta_{in}\beta_{in} + c_{44}\eta_{in}\gamma_{in}.
\end{aligned}
\tag{9}
$$

Hence, for each n, the surfaces $x_2 = \pm h$ are free of traction if

$$
\begin{aligned}
A_{1n}\lambda_{1n} \cos \eta_{1n}h + A_{2n}\lambda_{2n} \cos \eta_{2n}h + A_{3n}\lambda_{3n} \cos \eta_{3n}h &= 0,\\
A_{1n}\mu_{1n} \sin \eta_{1n}h + A_{2n}\mu_{2n} \sin \eta_{2n}h + A_{3n}\mu_{3n} \sin \eta_{3n}h &= 0,\\
A_{1n}\nu_{1n} \sin \eta_{1n}h + A_{2n}\nu_{2n} \sin \eta_{2n}h + A_{3n}\nu_{3n} \sin \eta_{3n}h &= 0,
\end{aligned}
\tag{10}
$$

from which the Ekstein dispersion relation is

$$
\begin{aligned}
E = {} & \lambda_{1n}(\mu_{2n}\nu_{3n} - \mu_{3n}\nu_{2n}) \cos \eta_{1n}h \sin \eta_{2n}h \sin \eta_{3n}h\\
& + \lambda_{2n}(\mu_{3n}\nu_{1n} - \mu_{1n}\nu_{3n}) \sin \eta_{1n}h \cos \eta_{2n}h \sin \eta_{3n}h\\
& + \lambda_{3n}(\mu_{1n}\nu_{2n} - \mu_{2n}\nu_{1n}) \sin \eta_{1n}h \sin \eta_{2n}h \cos \eta_{3n}h = 0
\end{aligned}
\tag{11}
$$

and the Ekstein displacement components are

$$
\begin{aligned}
u_1 &= \sum_{n=1}^{\infty}\sum_{i=1}^{3} D_n\zeta_{in} \cos \xi_n x_1 \sin \eta_{in} x_2,\\
u_2 &= \sum_{n=1}^{\infty}\sum_{i=1}^{3} D_n\zeta_{in}\beta_{in} \sin \xi_n x_1 \cos \eta_{in} x_2,\\
u_3 &= \sum_{n=1}^{\infty}\sum_{i=1}^{3} D_n\zeta_{in}\gamma_{in} \sin \xi_n x_1 \cos \eta_{in} x_2
\end{aligned}
\tag{12}
$$

in which $\zeta_{1n} = 1$ and $\zeta_{2n} = A_{2n}/A_{1n}$, $\zeta_{3n} = A_{3n}/A_{1n}$, i.e.

$$\zeta_{2n} = (\mu_{3n}\nu_{1n} - \mu_{1n}\nu_{3n}) \sin \eta_{1n}h/(\mu_{2n}\nu_{3n} - \mu_{3n}\nu_{2n}) \sin \eta_{2n}h,$$
$$\zeta_{3n} = (\mu_{1n}\nu_{2n} - \mu_{2n}\nu_{1n}) \sin \eta_{1n}h/(\mu_{2n}\nu_{3n} - \mu_{3n}\nu_{2n}) \sin \eta_{3n}h. \tag{13}$$

The Ekstein dispersion relation (11) has been studied in some detail in [5, 6].

3. FREQUENCIES OF A PLATE OF FINITE LENGTH

The components of traction on planes x_1 = constant are

$$T_{11} = c_{11}u_{1,1} + c_{12}u_{2,2} + c_{14}u_{3,2},$$
$$T_{12} = c_{56}u_{3,1} + c_{66}(u_{2,1} + u_{1,2}), \tag{14}$$
$$T_{13} = c_{55}u_{3,1} + c_{56}(u_{2,1} + u_{1,2}).$$

From (12) and (14),

$$T_{11} = -\sum_{n=1}^{\infty}\sum_{i=1}^{3} D_n\zeta_{in}a_{in} \sin \xi_n x_1 \sin \eta_{in}x_2,$$
$$T_{12} = \sum_{n=1}^{\infty}\sum_{i=1}^{3} D_n\zeta_{in}\lambda_{in} \cos \xi_n x_1 \cos \eta_{in}x_2, \tag{15}$$
$$T_{13} = \sum_{n=1}^{\infty}\sum_{i=1}^{3} D_n\zeta_{in}b_{in} \cos \xi_n x_1 \cos \eta_{in}x_2,$$

where

$$a_{in} = c_{11}\xi_n + c_{12}\eta_{in}\beta_{in} + c_{14}\eta_{in}\gamma_{in},$$
$$b_{in} = c_{56}\eta_{in} + c_{56}\xi_n\beta_{in} + c_{55}\xi_n\gamma_{in}. \tag{16}$$

We now restrict n to the first three branches of Ekstein's dispersion relation (11), i.e. the flexure, thickness-shear and face-shear branches, and set the resultant force and couple on $x_1 = \pm l$, per unit length along x_3, equal to zero. Thus, on $x_1 = \pm l$,

$$\int_{-h}^{h} x_2T_{11}\,dx_2 = 0, \quad \int_{-h}^{h} T_{12}\,dx_2 = 0, \quad \int_{-h}^{h} T_{13}\,dx_2 = 0, \tag{17}$$

from which

$$\sum_{n=1}^{3} D_nL_{1n} \sin \xi_n l = 0,$$
$$\sum_{i=1}^{3} D_nL_{2n} \cos \xi_n l = 0, \tag{18}$$
$$\sum_{i=1}^{3} D_nL_{3n} \cos \xi_n l = 0,$$

where

$$L_{1n} = \sum_{i=1}^{3} \zeta_{in} a_{in} \eta_{in}^{-2} (\sin \eta_{in} h - \eta_{in} h \cos \eta_{in} h),$$

$$L_{2n} = \sum_{i=1}^{3} \zeta_{in} \lambda_{in} \eta_{in}^{-1} \sin \eta_{in} h, \tag{19}$$

$$L_{3n} = \sum_{i=1}^{3} \zeta_{in} b_{in} \eta_{in}^{-1} \sin \eta_{in} h.$$

Upon eliminating the three D_n from (18), we obtain, finally, the equation governing the frequency spectrum of the plate of length $2l$ in the x_1-direction, thickness $2h$ in the x_2-direction and of infinite width in the x_3-direction:

$$\begin{aligned} F = {} & L_{11}(L_{22}L_{33} - L_{32}L_{23}) \sin \xi_1 l \cos \xi_2 l \cos \xi_3 l \\ & + L_{12}(L_{23}L_{31} - L_{33}L_{21}) \cos \xi_1 l \sin \xi_2 l \cos \xi_3 l \\ & + L_{13}(L_{21}L_{32} - L_{31}L_{22}) \cos \xi_1 l \cos \xi_2 l \sin \xi_3 l = 0. \end{aligned} \tag{20}$$

4. SOLUTION OF EQUATIONS

The equations to be solved are (5) and (11), for the Ekstein branches, and then (20) for the frequency spectrum. As both ξ_n and η_{in} appear in both (5) and (11) and, since (11) is transcendental, the two equations have to be solved simultaneously by successive approximations; and this must be done separately for each of the three branches for a given frequency ratio Ω. Trial values of the ξ_n, for the flexure and thickness-shear branches were found from an approximate equation obtained previously[7]:

$$\kappa^4 \bar{\xi}_n^4 + (1+g)\Omega^2 \kappa^2 \bar{\xi}_n^2 + g\Omega^2(\Omega^2 - 1) = 0, \quad n = 1,2, \tag{21}$$

where

$$\kappa^2 = \pi^2/12, \quad g = \kappa^2 c_{66}/(c_{11} - c_{12}^2/c_{22}). \tag{22}$$

For the trial values of $\bar{\xi}_3$, for the face-shear branch, the approximation

$$\bar{\xi}_3^2 = \Omega^2/\bar{c}_{55} \tag{23}$$

was employed.

With each of the three trial values of $\bar{\xi}_n$ and a chosen frequency ratio Ω, the three roots $\bar{\eta}_{in}^2$ of (5) were obtained and, along with their $\bar{\xi}_n$, employed to calculate a trial value of E from (11). Then the next approximation to $\bar{\xi}_n$ was computed from (5) and (11) by the secant method and the process, beginning with (5), was repeated until either E was less than a certain small quantity or passed through zero. In the latter case, the final value of $\bar{\xi}_n$ was computed by linear interpolation and the associated values of the $\bar{\eta}_{in}$ were finally obtained from (5). Thus, three $\bar{\xi}_n$ and three $\bar{\eta}_{in}$ for each of them were computed for the given frequency. The twelve quantities were then employed to compute the coefficients in (20) and to supply the $\xi_n h$ parts of the arguments $\xi_n l = (\xi_n h)(l/h)$. The remaining part, l/h, of the arguments was then increased, from a starting value, in small, equal increments until F passed through zero. Linear interpolation then gave an abscissa l/h on a branch of the frequency spectrum for the chosen frequency. l/h was then increased in successive, small increments until the next branch was encountered, as indicated by F again passing through zero. After covering the range of l/h desired, the next frequency was chosen and the entire procedure was repeated, beginning with trial values of the $\bar{\xi}_n$ from (21) and (23).

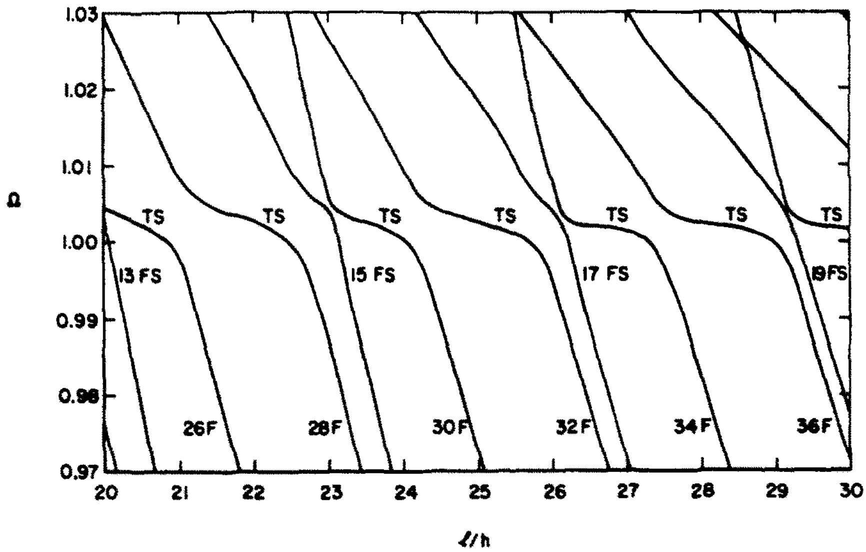

Fig. 1. Frequency spectrum of Flexure (F), Thickness-Shear (TS) and Face-Shear (FS) Modes of Vibration of AT-Cut Quartz Plates.

It should be noted that, from (5), in the range of the present application, the three $\bar{\eta}_{i1}$ for the flexure branch are imaginary; the three $\bar{\eta}_{i2}$ for the thickness-shear branch are real; while, for the face-shear branch, $\bar{\eta}_{13}$ and $\bar{\eta}_{33}$ are imaginary while $\bar{\eta}_{23}$ is real. Thus, the real form of (11) is different for each of the first three branches of Ekstein's equation.

In the function F, in (20), $\bar{\xi}_1$ and $\bar{\xi}_3$ are real but $\bar{\xi}_2$ is imaginary for $\Omega < 1$, zero for $\Omega = 1$ and real for $\Omega > 1$; so that F also takes three different real forms depending, in this case, on the frequency. It should be noted that, although $\sin \xi_2 l$ is zero when $\Omega = 1$, the entire second term in (20) does not then disappear. This is because of the presence of ξ_2 in the denominator of L_{12}.

The results of a typical computation are shown in Fig. 1 for the AT-cut of quartz for which, as calculated from Bechmann's constants[8]:

$$\bar{c}_{11} = 2.98969 \qquad \bar{c}_{12} = -0.284719$$

$$\bar{c}_{22} = 4.47269 \qquad \bar{c}_{14} = -0.125973$$

$$\bar{c}_{44} = 1.33083 \qquad \bar{c}_{24} = -0.196478$$

$$\bar{c}_{55} = 2.37159 \qquad \bar{c}_{56} = 0.0873254.$$

The ranges and intervals for the computation were

$$l/h\text{: } 20(0.1)30$$
$$\Omega\text{: } 0.97(0.0025)1(0.001)1.03.$$

About 150 min of computing time were required on a TRS-80 microcomputer to solve for the approximately 400 points on which Fig. 1 is based. The figure illustrates how the coupling of the thickness-shear modes with the face-shear modes can disturb the flat, thickness-shear frequency terraces as, for example, occurs in $22.5 < l/h < 24.1$ and in $25.7 < l/h < 27.3$ but not in $24.1 < l/h < 25.7$.

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